11 (a) Another solution is $\frac{1}{2}(x+X, y+Y, z+Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

12 Elimination leads to this upper triangular system; then comes back substitution.

$$\begin{aligned} 2x+3y+\ z&=8 \\ y+3z&=4 \\ 8z&=8 \end{aligned} \quad \text{gives} \quad \begin{aligned} x&=2 \\ y&=1 \\ z&=1 \end{aligned}$$

If a zero is at the start of row 2 or row 3, that avoids a row operation.

11 (a). $A\vec{x}=\vec{b}$ is the system.

$\vec{x}_1=\begin{pmatrix} x\\ y\\ z\end{pmatrix}$ $\vec{x}_2=\begin{pmatrix} X\\ Y\\ Z\end{pmatrix}$ are both solutions.

∵ $A\vec{x}_1=\vec{b}$. $A\vec{x}_2=\vec{b}$

∴ $A(p\vec{x}_1)=p\vec{b}$ (proved below).

$A((1-p)\vec{x}_2)=(1-p)\vec{b}$.

$\left(A(p\vec{x}_1)=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & & \\ a_{m1} & a_{m2} & a_{m3}\end{pmatrix}\begin{pmatrix} px\\ py\\ pz\end{pmatrix}\right.$

$=\begin{pmatrix} pa_{11}x+pa_{12}y+pa_{13}z \\ \cdots \\ pa_{m1}x+pa_{m2}y+pa_{m3}z\end{pmatrix}$

$=p\begin{pmatrix} a_{11}x+a_{12}y+a_{13}z \\ \cdots \\ a_{m1}x+a_{m2}y+a_{m3}z\end{pmatrix}$

$=p\begin{pmatrix} b_1 \\ \vdots \\ b_m\end{pmatrix}=p\vec{b}$.

∴ $A(p\vec{x}_1)+A((1-p)\vec{x}_2)=p\vec{b}+(1-p)\vec{b}=\vec{b}$.

∴ $A(p\vec{x}_1+(1-p)\vec{x}_2)=\vec{b}$.

∴ $p\vec{x}_1+(1-p)\vec{x}_2$ is the solution.

i.e. $p\begin{pmatrix} x\\ y\\ z\end{pmatrix}+(1-p)\begin{pmatrix} X\\ Y\\ Z\end{pmatrix}$ is the solution

$(\forall p\in\mathbb{R})$.

(b). ~~two~~ two points $\underbrace{\begin{pmatrix} x_1\\ y_1\\ z_1\end{pmatrix}}_{=\vec{x}_1}$, $\underbrace{\begin{pmatrix} x_2\\ y_2\\ z_2\end{pmatrix}}_{=\vec{x}_2}$.

∵ $A_{25\times3}\vec{x}_1=0$.

$A_{25\times3}\vec{x}_2=0$.

∴ $A_{25\times3}(c\vec{x}_1+d\vec{x}_2)=0$.

$\forall c,d\in\mathbb{R}$. $c\vec{x}_1+d\vec{x}_2$ is the points where they meet.

14 Subtract 2 times row 1 from row 2 to reach $(d-10)y-z=2$. Equation (3) is $y-z=3$. If $d=10$ exchange rows 2 and 3. If $d=11$ the system becomes singular.

15 The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. But equation (3) is the same so there is a *line of solutions* $(x,y,z)=(1,1,-1)$.

Date

12. $\begin{pmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ 20 \\ 0 \end{pmatrix}$.

$$\xrightarrow{①\times(-2)+②}\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}.$$

$$\xrightarrow{②\times 2+③}\begin{pmatrix} \underset{\sim}{2} & 3 & 1 \\ 0 & \underset{\sim}{1} & 3 \\ 0 & 0 & \underset{\sim}{8} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 8 \\ 4 \\ 8 \end{pmatrix}.$$

the pivots are circled by " ~ ".

$\therefore \begin{cases} z=1 \\ y=1 \\ x=2 \end{cases}$.

14. $\begin{pmatrix} 2 & 5 & 1 \\ 4 & d & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$.

$$\xrightarrow{①\times(-2)+②}\begin{pmatrix} 2 & 5 & 1 \\ 0 & d-10 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$

$\therefore$ when $d-10=0 \Leftrightarrow d=10$.
row exchange is needed.

$$\xrightarrow{\text{row exchange}}\begin{pmatrix} 2 & 5 & 1 \\ 0 & 1 & -1 \\ 0 & d-10 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

$$\xrightarrow{②\times(10-d)+③}\begin{pmatrix} 2 & 5 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & d-11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 3 \\ 32-3d \end{pmatrix}$$

when $d-11=0$. only 2 pivots
the system is singular

15. $\begin{pmatrix} 1 & b & 0 \\ 1 & -2 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\xrightarrow{①\times(-1)+②}\begin{pmatrix} 1 & b & 0 \\ 0 & -2-b & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

if $-2-b=0 \Leftrightarrow b=-2$
row exchange is needed

$$\xrightarrow{\text{row exchange}}\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 1 \\ 0 & -2-b & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{②\times(2+b)+③}\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1+b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ When $1+b=0 \Leftrightarrow b=-1$
there is a missing pivot

When $b=-1$.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

infinite solutions

$\begin{cases} x=-k \\ y=-k \\ z=k \end{cases} \xrightarrow{\text{example}} \begin{cases} x=1 \\ y=1 \\ z=-1 \end{cases}$

19 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular—no third pivot. Then if $t = 5$ the third equation is $0 = 0$ which allows infinitely many solutions. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1+2=$row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution. The three planes in the row picture form a triangular tunnel.

25 $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

19. $\begin{pmatrix} 1 & 4 & -2 \\ 1 & 7 & -6 \\ 0 & 3 & q \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ t \end{pmatrix}$.

①×(-1)+②

$\begin{pmatrix} 1 & 4 & -2 \\ 0 & 3 & -4 \\ 0 & 3 & q \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ t \end{pmatrix}$.

②×(-1)+③

$\begin{pmatrix} 1 & 4 & -2 \\ 0 & 3 & -4 \\ 0 & 0 & 4+q \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ t-5 \end{pmatrix}$.

∴ When $4+q=0 \Leftrightarrow q=-4$.
the system is singular.
At that time, if $t=5$,
there will be infinite solutions

When $z=1$

$\begin{pmatrix} 1 & 4 & -2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$.

$\Rightarrow \begin{cases} x=-9 \\ y=3 \\ z=1 \end{cases}$

20. ~~linear~~ combination.

$\begin{cases} x+y+z=0 \\ x-2y-z=1 \end{cases}$.

with $3y+2z=-1$ infinite solutions

with $3y+2z=-2$ no solutions

25. ① $a=0$ ✓

② When $a \neq 0$

$\begin{pmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{pmatrix}$.

$\begin{pmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{pmatrix}$.

∴ When $a=2$. ✓

$\begin{pmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & 0 & a-4 \end{pmatrix}$.

∴ When $a=4$. ✓