20200909 Problem Set 1.2

3. $\vec{v}_0 = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{5}\cdot\begin{pmatrix}4\\3\end{pmatrix}$

$= \begin{pmatrix}\frac{4}{5}\\ \frac{3}{5}\end{pmatrix}$.

$\vec{w}_0 = \frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{5}}\cdot\begin{pmatrix}1\\2\end{pmatrix}$

$= \begin{pmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix}$

$\vec{v}\cdot\vec{w} = \|\vec{v}\|\cdot\|\vec{w}\|\cdot\cos\theta$.

$4\times1 + 3\times2 = 5\cdot\sqrt{5}\cos\theta$.

$\cos\theta = \frac{2\sqrt{5}}{5}$

$\vec{a} = k_1\vec{w} \quad (k_1 > 0)$

$\vec{b}\cdot\vec{w} = 0 \quad (\vec{b} = \begin{pmatrix}b_1\\b_2\end{pmatrix})$

$b_1 + b_2\times2 = 0$

$b_1 = -2b_2$

$\therefore \vec{b}$ could be $\begin{pmatrix}-2t\\t\end{pmatrix}$.

$\vec{c} = k_2\vec{w} \quad (k_2 < 0)$.

4. (a) $\|\vec{v}\| = 1 \quad \therefore v_1^2 + v_2^2 = 1$

$\vec{v}\cdot(-\vec{v}) = -\vec{v}\cdot\vec{v}$

$= -(v_1^2 + v_2^2)$

$= -1$

(b) $(\vec{v}+\vec{w})(\vec{v}-\vec{w})$

$= \vec{v}\cdot\vec{v} - \vec{w}\cdot\vec{w}$

$= \|\vec{v}\|^2 - \|\vec{w}\|^2 = 0$.

(c) $(\vec{v}-2\vec{w})(\vec{v}+2\vec{w})$

$= \vec{v}\cdot\vec{v} - 4\vec{w}\cdot\vec{w}$

$= 1 - 4 = -3$.

7. (1). $\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$

$= \frac{1}{1\times2}$

$\therefore \theta = \frac{\pi}{3}$

(2) $\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$

$= \frac{4-2-2}{\|\vec{v}\|\|\vec{w}\|} = 0$

$\therefore \theta = 90^\circ = \frac{\pi}{2}$

(3) ~~$\cos\theta$~~ ~~$\theta = \pi$~~.

$\cos\theta = \frac{-1+3}{2\times2} = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$.

(4) $\cos\theta = \frac{-3-2}{\sqrt{10}\cdot\sqrt{5}} = -\frac{\sqrt{2}}{2}$

$\therefore \theta = \frac{3\pi}{4}$.

9. $\frac{v_2 w_2}{v_1 w_1} = -1$

$\therefore v_2 w_2 + v_1 w_1 = 0$

$\therefore \vec{v} \cdot \vec{w} = 0$.

$\therefore \vec{v} \perp \vec{w}$

13. $\begin{cases} \vec{v} \cdot \vec{w} = 0 \\ \vec{v} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \\ \vec{w} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \end{cases}$

$\therefore \begin{cases} v_1 w_1 + v_2 w_2 + v_3 w_3 = 0 \quad (*) \\ v_1 + v_3 = 0 \\ w_1 + w_3 = 0 \end{cases}$.

Set $v_1 = w_1 = 1$ $\therefore v_3 = -1$.
$w_3 = -1$.

$\therefore$ Input $_\wedge^{\text{them}}$ to $(*)$

$\therefore 1 \times 1 + v_2 w_2 + (-1) \times (-1) = 0$

$v_2 w_2 = -2$

$\therefore$ Set $v_2 = \sqrt{2}$, $w_2 = -\sqrt{2}$

$\therefore \vec{v} = \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ -1 \end{pmatrix}$.

14. $\begin{cases} u_1 + u_2 + u_3 + u_4 = 0 \\ v_1 + \cdots + v_4 = 0 \\ w_1 + \cdots + w_4 = 0 \quad (*) \end{cases}$

$\sum_{i=1}^{4} u_i v_i = \sum_{i=1}^{4} u_i w_i = \sum_{i=1}^{4} w_i v_i = 0$

Set $u_1 = u_2 = 1$,
$u_3 = u_4 = -1$

$\therefore \begin{cases} v_1 + v_2 - v_3 - v_4 = 0 \\ w_1 + w_2 - w_3 - w_4 = 0 \end{cases}$.

$\therefore \begin{cases} v_1 + v_2 = v_3 + v_4 \\ w_1 + w_2 = w_3 + w_4 \quad (1) \end{cases}$

Set $v_1 = v_3 = 1$, $v_2 = v_4 = -1$.

$\therefore w_1 - w_2 + w_3 - w_4 = 0$

$\therefore w_1 + w_3 = w_2 + w_4$. (2)

From (1). (2) (*).

~~if $w_1 = 1$~~. $\therefore w_2 = w_3$.
$w_1 = w_4$

$\therefore w_1 = w_4 = 1$.
$w_2 = w_3 = -1$.

$\therefore \vec{u} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$.

17. $\cos\alpha = \frac{1}{\sqrt{2}}$. $\Rightarrow \alpha = \frac{\pi}{4}$

$\cos\beta = 0$ $\quad \beta = \frac{\pi}{2}$

$\cos\theta = \frac{-1}{\sqrt{2}}$ $\quad \theta = \frac{3\pi}{4}$.

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\theta = 1$

27. $\|\vec{v}+\vec{w}\|^2+\|\vec{v}-\vec{w}\|^2$

$=(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})+(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$

$=2(\vec{v}\cdot\vec{v}+\vec{w}\cdot\vec{w})$.

$=2\|\vec{v}\|^2+2\|\vec{w}\|^2$

29. $\|\vec{v}-\vec{w}\|$

$=\sqrt{(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})}$

$=\sqrt{\|\vec{v}\|^2+\|\vec{w}\|^2-2\vec{v}\cdot\vec{w}}$

$=\sqrt{34-2\vec{v}\cdot\vec{w}}$

$\vec{v}\cdot\vec{w}=\|\vec{v}\|\|\vec{w}\|\cos\theta$

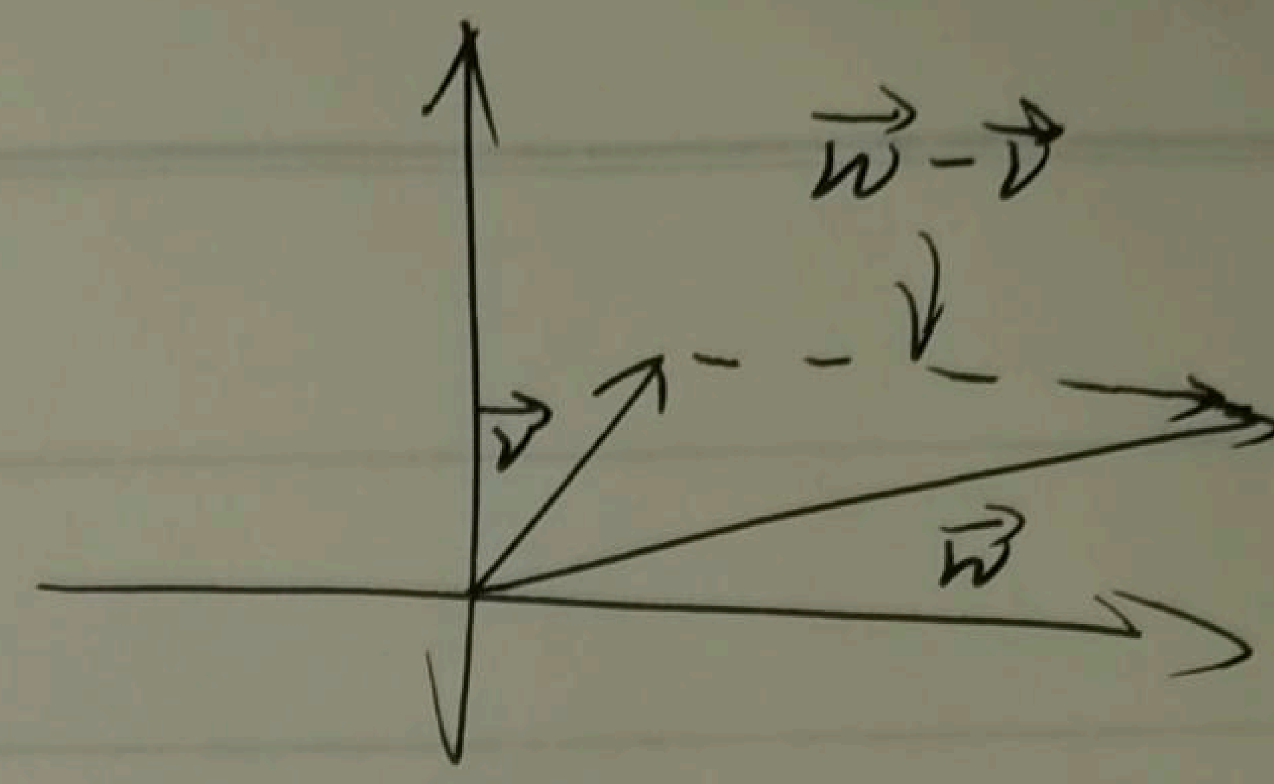
$=15\cos\theta$.

∴ $\vec{v}\cdot\vec{w}\in[-15,15]$

∴ $\|\vec{v}-\vec{w}\|\in[\sqrt{34-2\times15},\sqrt{34+2\times15}]$

∴ $2\le\|\vec{v}-\vec{w}\|\le 8$

$\vec{v}\cdot\vec{w}\in[-15,15]$



∵ Law of Cosine

∴ $\|\vec{w}-\vec{v}\|^2=\|\vec{v}\|^2+\|\vec{w}\|^2-2\|\vec{v}\|\|\vec{w}\|\cdot\cos\langle\vec{v}\cdot\vec{w}\rangle$

$(\vec{w}-\vec{v})\cdot(\vec{w}-\vec{v})=\vec{v}\cdot\vec{v}+\vec{w}\cdot\vec{w}-2\|\vec{v}\|\|\vec{w}\|\cos\langle\vec{v},\vec{w}\rangle$

$-2\vec{w}\cdot\vec{v}=-2\|\vec{v}\|\|\vec{w}\|\cos\langle\vec{v},\vec{w}\rangle$

∴ $\cos\langle\vec{v},\vec{w}\rangle=\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$.

∵ $|\cos\langle\vec{v},\vec{w}\rangle|\le 1$

∴ $\dfrac{|\vec{v}\cdot\vec{w}|}{\|\vec{v}\|\cdot\|\vec{w}\|}\le 1$

∴ $|\vec{v}\cdot\vec{w}|\le\|\vec{v}\|\cdot\|\vec{w}\|$

Schwarz Inequality:

$|\vec{v}\cdot\vec{w}|\le\|\vec{v}\|\cdot\|\vec{w}\|$.