Final Exam

Math6002

- You have 90 minutes to finish all the tasks
- Any plagiarism will lead to 0 score
- **Technology is allowed. But the process without tech is required for all tasks except for those noted by "Tech Allowed".**

1. (24points) $\boldsymbol{v_1} = (-1,2,1)^T$, $\boldsymbol{v_2} = (1,0,-2)^T$, $\boldsymbol{b} = (2,4,-1)^T$. Give
   a. the length of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$
   b. the angle between $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$
   c. the equation for the plane S spanned by $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$
   d. the normal vector of S
   e. the orthogonal complement of S
   f. the projection of $\boldsymbol{b}$ onto the plane S
2. (6points) Are these vectors $\boldsymbol{v_1} = (1,1,1,1)^T$, $\boldsymbol{v_2} = (2,3,2,4)^T$, $\boldsymbol{v_3} = (-2,1,2,5)^T$, and $\boldsymbol{v_4} = (-2,2,2,7)^T$ independent or dependent? Give the process.
3. (10points)
$$A = \begin{bmatrix} 1 & 6 & 3 \\ 2 & 7 & 4 \\ -1 & -1 & 3 \end{bmatrix}$$
   a. Give the inverse for A step by step
   b. (Tech Allowed) Solve the system of equation $A\boldsymbol{x} = \boldsymbol{b}$ in which $\boldsymbol{b} = (2,3,4)^T$
4. (10points) The **column space** of A is $x - 2y + z = 0$.
   a. Give a possible A
   b. Give the complete solution for $A\boldsymbol{x} = \boldsymbol{b}$ in which $\boldsymbol{b} = (2,3,-1)^T$
5. (10points) (Tech Allowed) $A = (\boldsymbol{v_1}, \boldsymbol{v_2})$, in which $v_1 = (1,2,\dots,100)^T$, $v_2 = \left(v_{2,1},\ v_{2,2}, \dots, v_{2,100}\right)^T$, $b = \left(1^{\frac{1}{3}}, \dots, 100^{\frac{1}{3}}\right)^T$
   a. If $v_{2,i} = 1$ for any $i = 1, 2, \dots, 100$, give the least squares solution for $Ax = b$
   b. If $v_{2,i} = 1 - 3i$ for any $i = 1, 2, \dots, 100$, what will happen? Explain why it happens.
6. (10points) For A
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & -3 \\ 0 & -3 & 2 \end{bmatrix}$$
   a. Give the LU factorization
   b. Give the LDU factorization
7. (20points) Give the basis for (1)column space, (2)nullspace (3) row space, and (4) left nullspace of the following matrix A
$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 5 \\ 2 & 3 & 3 & 7 & 8 \\ 0 & 2 & 2 & 2 & 4 \end{bmatrix}$$

8. We all know that the product AB could be understood as some linear combinations of all the columns of A. Based on this idea, prove that:
    a. $C(AB)$ must be a subspace of $C(A)$, in which $C(A)$ and $C(AB)$ are the column space of A and AB respectively.
    b. $C(AB) = C(A)$ when there exists a matrix C to ensure $ABC = A$ (Tip: $ABC = (AB)C$, which means that ABC contains linear combinations of AB)
9. (5points) A, B are both square matrices and $(I - A - B)$ is invertible. If $A^2 = A$ and $B^2 = B$, please use the matrix multiplication to prove that
    a. $rank(A) = rank(AB)$
    b. $rank(A) = rank(B)$