Test 1 for Unit 3.

1.

$$A = \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\therefore$ rank$(A) = 2$

$\therefore$ $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ ~~form~~ a basis for $C(A)$.

(1) for $N(A^T)$.

$\because N(A^T) \perp C(A)$.

(the orthogonal complement of $C(A)$).

$\therefore \forall \vec{x} \in N(A^T)$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$\therefore \vec{x}^T \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0. \quad \vec{x}^T \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0.$$

$$\therefore \begin{cases} x + y = 0 \\ 2x + 3y + z = 0 \end{cases}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}}_{B} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$B \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

$$\therefore \begin{cases} x - z = 0 \\ y + z = 0 \end{cases} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a basis for $N(A^T)$.

(2) $N(A)$.

$$A \rightarrow \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\therefore$ if $A\vec{x} = \vec{0}$. $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}$.

$$\therefore \begin{cases} x_1 + 2x_2 = 0 \\ x_3 + 2x_4 + 3x_5 = 0 \end{cases}.$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ -2x_4 - 3x_5 \\ x_4 \\ x_5 \end{pmatrix}.$$

$$= x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $N(A)$

(3) $C(A^T)$.

$$\because A \rightarrow \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\therefore$ the first two rows of A can be a basis

$\therefore (1, 2, 2, 4, 6)^T$ and $(1, 2, 3, 6, 9)^T$.

a. 2. ∴ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4\begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} + x_3\begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$.

$= \begin{pmatrix} -x_3+2x_4 \\ 2x_3+3x_4 \\ x_3 \\ x_4 \end{pmatrix}$.

∴ $\begin{cases} x_1 + x_3 - 2x_4 = 0 \\ x_2 - 2x_3 - 3x_4 = 0 \end{cases}$

∴ $A = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & -3 \end{pmatrix} = R$.

∴ basis for $C(A^T)$:

$(1,0,1,-2)^T$, $(0,1,-2,-3)^T$.

b 3. $(A;b) \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & -2 & -8 & 6 \\ 0 & 0 & 2 & 8 & -6 \end{array}\right)$.

$\to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.

$\to \left(\begin{array}{cccc|c} 1 & 2 & 0 & -4 & 7 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.

∴ $\vec{x}_s = \begin{pmatrix} 7 \\ 0 \\ -3 \\ 0 \end{pmatrix}$. for $A\vec{x} = \vec{b}$

∵ $R = \begin{pmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

∴ $\begin{cases} x_1 = -2x_2 + 4x_4 \\ x_3 = -4x_4 \end{cases}$ ∴ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 4 \\ 0 \\ -4 \\ 1 \end{pmatrix}$.

for $A\vec{x} = \vec{b}$

∴ $\vec{x} = \begin{pmatrix} 7 \\ 0 \\ -3 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 4 \\ 0 \\ -4 \\ 1 \end{pmatrix}$.

4. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y-3z \\ y \\ z \end{pmatrix}$

$= y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

∴ basis: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$.

The normal vector: $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

∴ ~~basis:~~ $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

4. (1) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x+3z \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.

∴ basis: $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.

(2) The normal vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

is the basis for the space we want.

5 a) $\begin{pmatrix} -2 & 1 & 5 \\ 3 & 0 & -6 \\ 4 & -1 & 10 \end{pmatrix}$.

$\rightarrow \begin{pmatrix} -2 & 1 & 5 \\ 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 20 \end{pmatrix}$

$\rightarrow \begin{pmatrix} -2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 1 & 20 \end{pmatrix}$

$\rightarrow \begin{pmatrix} -2 & 1 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\therefore$ Independent

b). There is $(0,0,0)^T$

$\therefore$ Dependent.

6. $\because A^TA\vec{x} = \vec{0}$.

$\therefore \vec{x}^TA^TA\vec{x} = 0$.

$\therefore (A\vec{x})^TA\vec{x} = 0$.

$\therefore A\vec{x} = \vec{0}$.

(Only the inner product of $\vec{0}$ and itself could be zero).

7. $A\underset{2\times1}{\vec{x}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

$\therefore A_{2\times2}$.

$\because \mathrm{Dim}(N(A)) = 1$.

$\therefore$ There are only one independent row in A.

$A = \begin{pmatrix} a_1 & a_2 \\ ka_1 & ka_2 \end{pmatrix}$.

$\because$ for $A\vec{x} = \vec{0}$, $\vec{x} = c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Which means

$x_1 = 0$.

or $x_1 + 0x_2 = 0$.

$\therefore a_1 = 1$, $a_2 = 0$.

$\therefore A = \begin{pmatrix} 1 & 0 \\ k & 0 \end{pmatrix}$.

$A\vec{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ has solution $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 & 0 \\ k & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

$\therefore k = 3$

$\therefore A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$.

8. $B = (\vec{\alpha}_1, \vec{\alpha}_2, \cdots \vec{\alpha}_N)$.

$\therefore AB = (A\vec{\alpha}_1, A\vec{\alpha}_2 \cdots A\vec{\alpha}_n)$.

$A\vec{\alpha}_i$ means the linear combination of columns of A

$\therefore$ all of combinations of $A\vec{\alpha}_1, A\vec{\alpha}_2 \cdots A\vec{\alpha}_n$ must be combination of columns of A.

$\therefore$ No. of independent columns of AB $\leq$ No. of - - - of A

i.e. $\text{rank}(AB) \leq \text{Rank}(A)$

$\because \text{rank}(A) = \text{rank}(A^T)$ for any A.

$\therefore \text{rank}((AB)^T) \leq \text{rank}(A^T)$.

i.e. $\text{rank}(B^T A^T) \leq \text{rank}(A^T)$.

set $B^T = A_{new}$ $A^T = B_{new}$.

$\therefore \text{rank}(A_{new} B_{new}) \leq \text{rank}(B_{new})$.