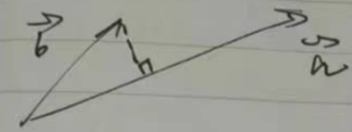
1. (a)



$\vec{p} = \hat{x}\vec{a}$

$\therefore (\vec{b} - \hat{x}\vec{a})^T \vec{a} = 0$

$\vec{b}^T\vec{a} - \hat{x}\cdot(\vec{a}^T\vec{a}) = 0$

$\therefore \hat{x} = \dfrac{\vec{b}^T\vec{a}}{\vec{a}^T\vec{a}}$.

$$\therefore \vec{p} = \frac{\vec{b}^T\vec{a}}{\vec{a}^T\vec{a}}\vec{a} = \frac{1+2+2}{1+1+1}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{5}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

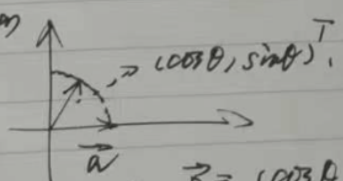
$$\vec{e} = \vec{b} - \vec{p} = \begin{pmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{pmatrix}$$

$$\therefore \vec{e}^T\vec{a} = -\frac{2}{3}\times1 + \frac{1}{3}\times1 + \frac{1}{3}\times1 = 0.$$

(b). $\vec{p} = \dfrac{\vec{b}^T\vec{a}}{\vec{a}^T\vec{a}}\vec{a} = \dfrac{-1-9-1}{1+9+1}\vec{a} = -\vec{a}$

$\therefore \vec{e} = \vec{b} - \vec{p} = \vec{b} + \vec{a} = \vec{0}$.

2. (a)

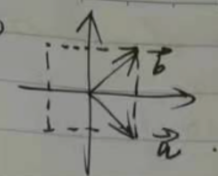


$\therefore \vec{p} = (\cos\theta, 0)^T$.

Also, $\vec{p} = \dfrac{\vec{b}^T\vec{a}}{\vec{a}^T\vec{a}}\vec{a}$

$$= \frac{\cos\theta}{1}\vec{a} = \begin{pmatrix}\cos\theta\\0\end{pmatrix}.$$

(b)



$\therefore \vec{p} = \vec{0}$

Also. $\vec{p} = \dfrac{\vec{b}^T\vec{a}}{\vec{a}^T\vec{a}}\vec{a} = \dfrac{0}{\vec{a}^T\vec{a}}\vec{a} = \vec{0}$.

3. $$P_1 = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{3}(1\ 1\ 1)^T(1\ 1\ 1) = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}.$$

$$P_2 = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{11}\begin{pmatrix}-1\\-3\\-1\end{pmatrix}(-1\ -3\ -1) = \frac{1}{11}\begin{pmatrix}1&3&1\\3&9&3\\1&3&1\end{pmatrix}.$$

$$P_1\vec{b} = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}5\\5\\5\end{pmatrix}.$$

$$P_2\vec{b} = \frac{1}{11}\begin{pmatrix}1&3&1\\3&9&3\\1&3&1\end{pmatrix}\begin{pmatrix}1\\3\\1\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix}.$$

$$P_1^2 = \frac{1}{9}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} = \frac{1}{9}\begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} = P_1$$

$$P_2^2 = \frac{1}{121}\begin{pmatrix}1&3&1\\3&9&3\\1&3&1\end{pmatrix}\begin{pmatrix}1&3&1\\3&9&3\\1&3&1\end{pmatrix} = \frac{1}{121}\begin{pmatrix}11&33&11\\33&99&33\\11&33&11\end{pmatrix} = \frac{1}{11}\begin{pmatrix}1&3&1\\3&9&3\\1&3&1\end{pmatrix} = P_2.$$

4. $$P_1 = \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) = \begin{pmatrix}1&0\\0&0\end{pmatrix}.$$

$$P_2 = \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(1\ -1) = \frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}.$$

$P_1 = A(A^TA)^{-1}A^T$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$P_1 \cdot \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ ✓

~~$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ form a basis of $C(A)$. (xy plane)~~

$P_2 = A(A^TA)^{-1}A^T$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$P_2^2 = P_2$ easy to be verified.

13. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

The projection must also be 4 by 1.

$\therefore P_{4\times4} \vec{b}_{4\times1} = \vec{p}_{4\times1}$

$\therefore P_{4\times4}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \vec{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$$

15. A.

$\hat{x}$ for $2A$ should be <u>half</u> of $\hat{x}$ for A.

17. $(I-P)^2 = I^2 + P^2 - 2P$

$= I + P - 2P = I - P$.

$\vec{p} = P\vec{b} \in C(A)$

$\therefore \vec{e} = \vec{b} - \vec{p}$, $\vec{e} \perp C(A)$.

$\because N(A^T) = [C(A)]^{\perp}$

$\therefore \vec{e} \in N(A^T)$.

$\therefore \vec{e} = \vec{b} - \vec{p} = \vec{b} - P\vec{b}$

$= (I-P)\vec{b} \in N(A^T)$

$\therefore I-P$ projects onto the $N(A^T)$.

19. $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $A_0 = (1 \ -1 \ -2)$

$\therefore$ the plane: $A_0\vec{x} = \vec{0}$.

$\therefore$ the plane is the nullspace of A_0, i.e. $N(A_0)$.

basis for $N(A_0)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+2z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ basis: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\therefore A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then the plane is also the $C(A)$

$\therefore P = A(A^TA)^{-1}A^T$

$$= \begin{pmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

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∴ $P_1+P_2 = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

$$(P_1+P_2)^2 = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & -1 \\ -1 & \frac{1}{2} \end{pmatrix} \neq P_1+P_2$$

(∵ $P_1^2 = P_1$ $\quad P_2^2 = P_2$.

∴ $(P_1+P_2)^2 = P_1^2 + P_1P_2 + P_2P_1 + P_2^2$

if $(P_1+P_2)^2 = P_1+P_2$

∴ $P_1P_2 + P_2P_1 = 0$ (*)

But (*) is not true) ∴

5. $P_1 = \dfrac{\vec{a}_1\vec{a}_1^T}{\vec{a}_1^T\vec{a}_1} = \dfrac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$.

$P_2 = \dfrac{\vec{a}_2\vec{a}_2^T}{\vec{a}_2^T\vec{a}_2} = \dfrac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$.

$P_1P_2 = \dfrac{1}{81}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$.

∵ $\vec{a}_1^T\vec{a}_2 = 0$

∴ $\vec{a}_1 \perp \vec{a}_2$

P_1P_2 means an operation:

step 1: Projection onto $\vec{a}_2$

step 2: Projection onto $\vec{a}_1$ after step 1

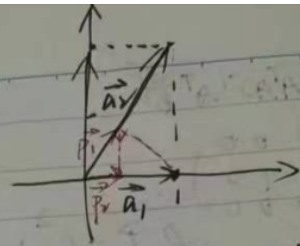
∵ $\vec{a}_1 \perp \vec{a}_2$

∴ $\forall \vec{v}$ after this operation will lead to $\vec{0}$.

∴ $\vec{0} = P_1P_2\vec{v}$ for $\forall \vec{v}$

∴ $P_1P_2 = 0$.

(b).



$P_1 = \dfrac{\vec{a}_1\vec{a}_1^T}{\vec{a}_1^T\vec{a}_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$P_2 = \dfrac{\vec{a}_2\vec{a}_2^T}{\vec{a}_2^T\vec{a}_2} = \dfrac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{pmatrix}$.

∴ $P_1P_2 = \begin{pmatrix} 0.2 & 0.4 \\ 0 & 0 \end{pmatrix}$.

∴ $P_1P_2\vec{a}_1 = \begin{pmatrix} 0.2 & 0.4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}$

∵ $(P_1P_2)^2 = \begin{pmatrix} 0.2 & 0.4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0.2 & 0.4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.8 \\ 0 & 0 \end{pmatrix} \neq P_1P_2$.

Also $(P_1P_2)^T \neq P_1P_2$

∴ Not projection.

11. $A^TA\hat{x} = A^T\vec{b}$.

$\hat{x} = (A^TA)^{-1}A^T\vec{b}$.

(1) ∴ $\vec{p} = A(A^TA)^{-1}A^T\vec{b} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ (Desmos / Excel can be used)

$\vec{e} = \vec{b} - \vec{p} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$. ∴ $\vec{e}^TA = (0,0)$

(2) $\vec{p} = A(A^TA)^{-1}A^T\vec{b} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$

∴ $\vec{e} = \vec{b} - \vec{p} = \vec{0}$. ($\vec{b} \in C(A)$).

20. The normal vector of the plane : $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \vec{e}$.

$$Q = \frac{\vec{e}\vec{e}^T}{\vec{e}^T\vec{e}} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

$\therefore P = I - Q$.

($\because$ Q can be also understood as a projection matrix onto the orthogonal complement of $C(A)$ in (19). i.e. $N(A^T)$).

21. $P^2 = [A(A^TA)^{-1}A^T][A(A^TA)^{-1}A^T]$
$= A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T$
$= A(A^TA)^{-1}A^T = P$.

$\therefore P(P\vec{b}) = P^2\vec{b} = P\vec{b}$.

itself

(answer ~~for~~ the ~~[illegible]~~ blank).

22. $P^T = [A(A^TA)^{-1}A^T]^T$
$= (A^T)^T[(A^TA)^{-1}]^TA^T$

$(A^T)^T = A$.

$[(A^TA)^{-1}]^T = [(A^TA)^T]^{-1}$
$= (A^TA)^{-1}$

$\therefore P^T = A(A^TA)^{-1}A^T = P$.

23. When A is invertible
$\therefore A_{n\times n}$ is formed by n basis of R^n.

$\therefore$ it means we want to project a vector $\vec{b} \in R^n$ onto R^n, which means no further things need to be done

$\therefore P = I$. $\vec{e} = \vec{0}$.

24. orthogonal.

$\because A^T\vec{b} = \vec{0}$

$\therefore \vec{b} \in N(A^T)$.

$\therefore \vec{b} \perp C(A)$.

$\therefore \vec{p} = \vec{0}$.

$\therefore P\vec{b} = A(A^TA)^{-1}A^T\vec{b}$
$= A(A^TA)^{-1}(A^T\vec{b})$

$\because A^T\vec{b} = \vec{0}$. $\therefore P\vec{b} = \vec{0}$.

25. When every possible $\vec{b} \in R^m$ has been projected, the $P\vec{b}$s must fill the whole S.

$\because P\vec{b}$ means the linear combination of the columns of P.

$\therefore S = C(P)$.

$\because Dim(S) = n$

$\therefore Dim(C(P)) = n$

$\therefore rank(P) = n$.

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26. $\because$ A invertible

$\therefore A^{-1}A^2 = A^{-1}A$

$(A^{-1}A)A = I$

$A = I$

27. $\because A^TA\vec{x} = \vec{0}$

$\therefore A^T(A\vec{x}) = \vec{0}$

$\therefore A\vec{x} \in N(A^T)$

Also. $A\vec{x} \in C(A)$.

$\because N(A^T) = [C(A)]^{\perp}$

$\therefore N(A^T) \cap C(A) = \vec{0}$.

$\therefore$ if $A\vec{x} \in N(A^T)$ and $A\vec{x} \in C(A)$

$\therefore A\vec{x} = \vec{0}$.

34. ① When P_1P_2 is a projection matrix.

$\therefore (P_1P_2)^T = P_1P_2$

$\because (P_1P_2)^T = P_2^TP_1^T$

and $P_2^T = P_2$ $P_1^T = P_1$

$\therefore (P_1P_2)^T = P_2^TP_1^T$

$= P_2P_1$

$\therefore P_1P_2 = P_2P_1$.

② When $P_1P_2 = P_2P_1$.

all the process in ① can be reversed.

$\therefore (P_1P_2)^T = P_1P_2$

$\therefore$ projection matrix .