Problem Set 3.5

1. (a) ∴ $A_{3\times 2}$

Try $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

A basis for $C(A^T) = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$.

it's obvious that

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is in the $C(A^T)$.

(b). ~~∴ $A_{3\times 3}$~~

∵ $A\vec{x} = \vec{0}$
$\quad\ \ _{3\times 1}$

∴ $A_{3\times 3}$.

∴ $Dim(C(A)) + Dim(N(A)) = 3$.

But now

$Dim(C(A)) = Dim(N(A)) = 1$.

∴ Impossible.

(c). $A_{m\times n}$.

if $Dim(N(A)) = r$.

∴ $Dim(C(A)) = n-r$

∴ $Dim(C(A^T)) = n-r$.

∴ $Dim(N(A^T)) = m-(n-r)$.

∴ $r = 1 + m-(n-r)$.

$0 = 1 + m - n$.

$n-1 = m$.

∴ for example $A = (1\ 2)$.

(d). $A_{2\times 2}$.

∵ $Dim(C(A)) \geq 1$. $Dim(N(A)) \geq 1$

and $Dim(C(A)) + Dim(N(A)) = 2$

∴ $Dim(C(A)) = Dim(N(A)) = 1$.

For example

∴ $A = \begin{pmatrix} 3 & 3a \\ 1 & a \end{pmatrix}$.

~~∴ $(A)^T$~~

$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$.

∴ $\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \vec{0}$

∴ $1 + 3a = 0$

$a = -\frac{1}{3}$.

∴ $A = \begin{pmatrix} 3 & -1 \\ 1 & -\frac{1}{3} \end{pmatrix}$.

(e). ∴ $A_{n\times n}$.

$Dim(N(A)) = n-r$

$Dim(N(A^T)) = n-r$.

∴ $N(A)$ and $N(A^T)$ have the same dimension.

Also, ① $N(A) \perp C(A^T)$.

and $N(A)$ is the "complement" of $C(A^T)$ for R^n.

② $N(A^T) \perp C(A)$

and $N(A^T)$ is the "complement" of $C(A)$ for R^n.

∵ $C(A) = C(A^T)$

∴ the complements are also the same ∴ $N(A^T) = N(A)$.

[complement: a basis of $N(A^T)$ and a basis of $C(A)$ span the R^n]

5. $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$.

$\vec{v_1} = (1,1,1)^T$ $\quad \vec{v_2} = (2,1,0)^T$.

$\because \begin{cases} B\vec{v_1} = \vec{0} \\ B\vec{v_2} = \vec{0} \end{cases}$

$\therefore \vec{v_2} - \vec{v_1} = (1, 0, -1)^T$

is also ~~in~~ the $N(B)$.

$\therefore \dashrightarrow \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ in the $N(B)$.

shape of "identity matrix"

$\therefore \vec{x} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$ $\quad \therefore x_1 - 2x_2 + x_3 = 0$

$\therefore (1\ -2\ 1) = B$.

6. No. of pivots $= 2$.

$\therefore \mathrm{Dim}(C(A)) = \mathrm{Dim}(C(A^T)) = 2$

$\therefore$ Row space $= \left\{ \vec{v} \,\middle|\, \vec{v} = a_1 \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \forall a_1, a_2 \in R \right\}$

Column space $= \left\{ \vec{v} \,\middle|\, \vec{v} = b_1 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}$ $b_1, b_2 \in R$.

$\mathrm{Dim}(N(A)) = 4 - 2 = 2$

$\mathrm{Dim}(N(A^T)) = 3 - 2 = 1$.

for any $\vec{x} \in N(A)$ $\quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$.

x_1, x_4 is free., $\begin{cases} x_3 = 0 \\ x_2 + x_4 = 0 \end{cases}$

$\therefore N(A) = \left\{ \vec{x} \,\middle|\, \vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\forall x_1, x_4 \in R$.

for any $\vec{y} \in N(A^T)$.

$\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \vec{0}$.

$\therefore y_2$ is free.

$3y_1 + y_3 = 0$

$3y_1 = 0$

$\therefore N(A^T) = \left\{ \vec{y} \,\middle|\, \vec{y} = y_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

9. ① ~~No. of~~ Independent rows in (A) and $\begin{pmatrix} A \\ A \end{pmatrix}$ must be the same.

$\therefore C(A^T) = C\left(\begin{pmatrix} A \\ A \end{pmatrix}^T\right)$.

$\mathrm{rank}(A^T) = \mathrm{rank}\left(\begin{pmatrix} A \\ A \end{pmatrix}^T\right) = r$

$\because N(A)$ is the "complement" of $C(A^T)$.

$N\begin{pmatrix} A \\ A \end{pmatrix}$ -- $C\left(\begin{pmatrix} A \\ A \end{pmatrix}^T\right)$.

$\therefore N(A) = N\begin{pmatrix} A \\ A \end{pmatrix}$

② Independent columns in $\begin{pmatrix} A \\ A \end{pmatrix}$ and $\begin{pmatrix} A & A \\ A & A \end{pmatrix}$ must be the same.

$\therefore C(A) = C\begin{pmatrix} A & A \\ A & A \end{pmatrix}$

$\therefore N(A^T) = N\left[\begin{pmatrix} A & A \\ A & A \end{pmatrix}^T\right]$

11. $\because$ $rank(A) \neq rank(A\ b)$.
if $m \le n$.
$rank(A)$ must be the same with $rank(A\ b)$.
$\therefore$ $m > n$.
~~$\therefore$ r≤~~

11. (a) for $rref(A \vdots b)$.
only when there is a row like
$(0\ 0 \cdots 0 \vdots a)$
$a \neq 0$.
$A\vec{x} = \vec{b}$ has no solution
$\therefore$ for the rref (A)
there must be at least one row $(0\ 0 \cdots 0)$.
$\therefore$ $r < m$.
also $r \le n$.
but for m, n, it's impossible to compare them

method 1
(b) $\because$ $A = \begin{pmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_m^T \end{pmatrix}$,

$\therefore$ $A^T\vec{y} = (\vec{\alpha}_1 \vec{\alpha}_2 \cdots \vec{\alpha}_m) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$,

$= y_1\vec{\alpha}_1 + y_2\vec{\alpha}_2 + \cdots + y_m\vec{\alpha}_m$.

$\because$ there is at least $(0 \cdots 0)$ in the rref (A)
$\therefore$ there must $\exists b_1, b_2 \cdots b_m \in R$
s.t. $b_1\vec{\alpha}_1^T + b_2\vec{\alpha}_2^T + \cdots + b_m\vec{\alpha}_m^T = \vec{0}^T$

$\therefore$ $b_1\vec{\alpha}_1 + b_2\vec{\alpha}_2 + \cdots + b_m\vec{\alpha}_m = \vec{0}$
it's obvious $b_1, b_2 \cdots b_m$ can't all be 0 in our discussion
($\because$ we're running the elimination)
choose one $b_i \neq 0$
$\therefore$ $b_i\vec{\alpha}_i = -(b_1\vec{\alpha}_1 + b_2\vec{\alpha}_2 + \cdots b_{i-1}\vec{\alpha}_{i-1} + b_{i+1}\vec{\alpha}_{i+1} + \cdots + b_m\vec{\alpha}_m)$,

$\therefore$ if $\vec{0} = A^T\vec{y} = y_1\vec{\alpha}_1 + \cdots + y_m\vec{\alpha}_m$.

Then $A^T\vec{y} = y_1\vec{\alpha}_1 + \cdots + y_{i-1}\vec{\alpha}_{i-1} + y_i(-b_1\vec{\alpha}_1 - b_2\vec{\alpha}_2 - \cdots - b_m\vec{\alpha}_m) + y_{i+1}\vec{\alpha}_{i+1} + \cdots + y_m\vec{\alpha}_m$,

$= (y_1 - b_1y_i)\vec{\alpha}_1 + (y_2 - b_2y_i)\vec{\alpha}_2 + \cdots + (y_{i-1} - b_{i-1}y_i)\vec{\alpha}_{i-1} + (y_{i+1} - b_{i+1}y_i)\vec{\alpha}_{i+1} + \cdots + (y_m - b_my_i)\vec{\alpha}_m$.

When $y_1 = b_1y_i$ & $y_2 = b_2y_i$ $\cdots$ $y_m = b_my_i$
this $(y_1 \cdots y_m)^T$ must be a solution.
(y_i is free).

Method 2:
$\because$ $m > r$
$\therefore$ $Dim(N(A^T)) = m - r > 0$
$\therefore$ must contain nonzero vector

No.

Date $A_{3\times3}$ rank$(A)=2$

12. $\vec{\alpha}_1=(1,0,1)^T$
$\vec{\alpha}_2=(1,2,0)^T$
$\because \vec{\alpha}_1,\vec{\alpha}_2$ is the basis for $C(A)$
$\therefore A=\underbrace{(\vec{\alpha}_1\ \vec{\alpha}_2)}_{3\times2}\begin{pmatrix}\vec{\beta}_1^T\\ \vec{\beta}_2^T\end{pmatrix}$
$\vec{\beta}_1,\vec{\beta}_2: 3\times2$
$\because \vec{\alpha}_1,\vec{\alpha}_2$ is the basis for $C(A^T)$.
$\therefore A=(\vec{\gamma}_1\ \vec{\gamma}_2)\begin{pmatrix}\vec{\alpha}_1^T\\ \vec{\alpha}_2^T\end{pmatrix}$
$\vec{\gamma}_1,\vec{\gamma}_2: 3\times2$.
$\therefore$ A obvious example for A can be
$(\vec{\alpha}_1\ \vec{\alpha}_2)\begin{pmatrix}\vec{\alpha}_1^T\\ \vec{\alpha}_2^T\end{pmatrix}=\begin{pmatrix}2&2&1\\2&4&0\\1&0&1\end{pmatrix}$.
$\text{Dim}(C(A^T))+\text{Dim}(N(A^T))=3$.
but now
$\text{Dim}(C(A^T))=2=\text{Dim}(N(A^T))$.

14. $A=\begin{pmatrix}\vec{\alpha}_1^T\\ \vdots\\ \vec{\alpha}_m^T\end{pmatrix}$.
$A^T\vec{y}=(\vec{\alpha}_1\ \vec{\alpha}_2\cdots\vec{\alpha}_m)\begin{pmatrix}y_1\\ \vdots\\ y_m\end{pmatrix}$
$\therefore \vec{b}=y_1\vec{\alpha}_1\cdots+y_m\vec{\alpha}_m$
$\therefore \vec{b}\in C(A^T)$.
When $N(A^T)=Z$.
$A^T\vec{y}=\vec{d}$ has no $\vec{x}_n\neq\vec{0}$.
to let
$A^T(\vec{y}+\vec{x}_n)=\vec{d}$.

25. (a) ✓
(b) ✗. $A=(1\ 0)$. $A^T=\begin{pmatrix}1\\0\end{pmatrix}$
$N(A^T)\neq N(A)=\{\vec{x}\mid\vec{x}=x\begin{pmatrix}0\\1\end{pmatrix}\}$.
$N(A^T)=Z$.
(c) When A^{-1} exists
$C(A)=C(A^T)=R^n$
but A can be unsymmetric
(d) ✓ $\because A^T=-A$
$\therefore m=n$
$A=\begin{pmatrix}\vec{\alpha}_1^T\\ \vdots\\ \vec{\alpha}_n^T\end{pmatrix}$ $A^T=(\vec{\alpha}_1\ \vec{\alpha}_2\cdots\vec{\alpha}_n)$
$-A=\begin{pmatrix}-\vec{\alpha}_1^T\\ -\vec{\alpha}_2^T\\ \vdots\\ -\vec{\alpha}_n^T\end{pmatrix}$
$\therefore \vec{\alpha}_i=\vec{\beta}_i$
~~$\therefore C(A^T)=$~~
A^T ✓
$A=\begin{pmatrix}0&a_{11}&a_{12}&\cdots&a_{1n}\\-a_{11}&0&a_{23}&\cdots&a_{2n}\\ \vdots&-a_{23}&\ddots&&\\ -a_{1n}&a_{2n}&&&0\end{pmatrix}=\begin{pmatrix}\vec{\alpha}_1^T\\ \vec{\alpha}_2^T\\ \vdots\\ \vec{\alpha}_n^T\end{pmatrix}=(\vec{\beta}_1\cdots\vec{\beta}_n)$

13. (a) ✗ usually different
(b) ✓
(c). ✗ for example
when $A_{n\times n}$ is invertible
$B_{n\times n}$ is ( )
$C(A)=C(A^T)=C(B)=C(B^T)$
$=R^n$
$N(A)=N(A^T)=N(B)=N(B^T)=Z$

15. $C(A^T)$. $N(A)$ stay the same.
$(2,1,3,4)^T$.

16. $\therefore \vec{v}\in C(A^T)$. $\vec{v}\in N(A)$
$\because C(A^T)\perp N(A)$
$\therefore \vec{v}^T\vec{v}=0$.

11. ∵ ...
if ...
11. ∵ ...