20221122 Assignment.

15. All solutions for $A\vec{x}=\vec{0}$ can be written as

$$\vec{x} = x_3\begin{pmatrix}2\\2\\1\\0\end{pmatrix} + x_4\begin{pmatrix}3\\1\\0\\1\end{pmatrix} = \begin{pmatrix}2x_3+3x_4\\2x_3+x_4\\x_3\\x_4\end{pmatrix}$$

$$\therefore \begin{cases} x_1 = 2x_3+3x_4 \\ x_2 = 2x_3+x_4 \\ x_3 \\ x_4 \end{cases} \text{free}$$

$\therefore$ $x_1-2x_3-3x_4=0$, $x_2-2x_3-x_4=0$ is $A\vec{x}=\vec{0}$

$\therefore$ A can be

$$\begin{pmatrix}1 & 0 & -2 & -3\\ 0 & 1 & -2 & -1\end{pmatrix}.$$

16. All solutions for $A\vec{x}=\vec{0}$ can be written as

$$\vec{x} = x_4\begin{pmatrix}4\\3\\2\\1\end{pmatrix} = \begin{pmatrix}4x_4\\3x_4\\2x_4\\x_4\end{pmatrix}$$

$$\therefore \begin{cases} x_1 = 4x_4 \\ x_2 = 3x_4 \\ x_3 = 2x_4 \\ x_4 \ \text{free} \end{cases}$$

$$\therefore A = \begin{pmatrix}1 & 0 & 0 & -4\\ & 1 & 0 & -3\\ 0 & & 1 & -2\end{pmatrix}.$$

$\therefore$ $\text{rank}(A) = 3$.

17. $$\begin{pmatrix}1 & 0 & a\\ 1 & 3 & b\\ 5 & 1 & c\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\therefore \begin{cases} 1+2a=0 \\ 1+3+2b=0 \\ 5+1+2c=0 \end{cases} \quad \begin{matrix} a=-\frac{1}{2} \\ b=-2 \\ c=-3 \end{matrix}$$

$$\therefore \begin{pmatrix}1 & 0 & -\frac{1}{2}\\ 1 & 3 & -2\\ 5 & 1 & -3\end{pmatrix}.$$

18. $\because$ Dimension of $C(A) \geq 2$
Dimension of $N(A) \geq 2$
$\therefore$ $n \geq 2+2=4$.
$\because$ At least 4 columns in A
$\therefore$ must be at least 4 unknowns to be multiplied

But, there are just 3 unknowns.
$\therefore$ Impossible.

24 $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ shows that (a)(b)(c) are all false. Notice $\text{rref}(A^{\mathrm{T}}) = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$.

27. <u>nullspace</u>.

$\because$ for $\forall\ \vec{\beta_i} \in C(B)$.

$AB = 0$,

$\Rightarrow A(\vec{\beta_1}\ \vec{\beta_2} \cdots \vec{\beta_n}) = 0$

$(A\vec{\beta_1}\ A\vec{\beta_2} \cdots A\vec{\beta_n}) = 0$

$\therefore \forall\ A\vec{\beta_i} = 0$

$\therefore \vec{\beta_i} \in N(A)$.

28.
$$C = \begin{pmatrix} A_{m_1 \times n} \\ B_{m_2 \times n} \end{pmatrix}.$$

for $C\vec{x} = \vec{0}$, it can be written as

$$\begin{pmatrix} A \\ B \end{pmatrix} \vec{x}_{n\times 1} = \vec{0} \Leftrightarrow \begin{pmatrix} A\vec{x} \\ B\vec{x} \end{pmatrix} = \vec{0}$$

$\therefore A\vec{x} = \vec{0}$ and $B\vec{x} = 0$.

$\therefore C\vec{x} = \vec{0} \Leftrightarrow \begin{cases} A\vec{x} = 0 \text{ and} \\ B\vec{x} = 0. \end{cases}$

$\therefore \vec{x}$ is in ~~C(A)~~ $N(A)$ and $N(B)$.

$\therefore \vec{x} \in N(A) \cap N(B)$.

(? is $N(A) \cap N(B)$ a space ? :

set $\forall\ \vec{x_1}, \vec{x_2} \in N(A) \cap N(B)$.

$\therefore A\vec{x_1} = 0$, $A\vec{x_2} = 0$

$B\vec{x_1} = 0$, $B\vec{x_2} = 0$

$\therefore A(c_1\vec{x_1} + c_2\vec{x_2}) = 0$

$B(c_1\vec{x_1} + c_2\vec{x_2}) = 0$. $(\forall c_1, c_2)$.

$\therefore c_1\vec{x_1} + c_2\vec{x_2} \in N(A) \cap N(B)$

$\therefore N(A) \cap N(B)$ is a space.

29. $\because A_{4\times 4}$ is invertible

$\therefore$ ~~N(A)~~ $A\vec{x} = \vec{0}$

has only solution $\vec{x} = \vec{0}$.

$\therefore N(A) = Z$.

for $B_{4\times 8} = (A\ A)$

$B\vec{x} = \vec{0}$ can be written as

$$(A\ A)\begin{pmatrix} \vec{x_1} \\ \vec{x_2} \end{pmatrix} = \vec{0}$$

in which $\vec{x_i}$s all both $4\times$

$\therefore A\vec{x_1} + A\vec{x_2} = \vec{0}$.

$A(\vec{x_1} + \vec{x_2}) = \vec{0}$.

$\therefore \vec{x_1} + \vec{x_2}$ must be $\vec{0}_{(4\times 1)}$

$\therefore \begin{pmatrix} \vec{x_1} \\ \vec{x_2} \end{pmatrix}$ could be written as

$$\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \end{pmatrix} = \begin{pmatrix} -x_{12} \\ -x_{22} \\ -x_{32} \\ -x_{42} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \end{pmatrix}$$

in which $x_{12}, x_{22}, x_{32}, x_{42}$ are all free.

$\therefore N(A)$ is spanned by 4 independent vectors

$\therefore N(A) = \mathbb{R}^4$.