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1. $AB = A(\vec{\beta_1}, \vec{\beta_2} \cdots \vec{\beta_k})$.

$= (A\vec{\beta_1}, A\vec{\beta_2} \cdots A\vec{\beta_k})$

$A\vec{\beta_i} = (\vec{\alpha_1}, \vec{\alpha_2} \cdots \vec{\alpha_n})\begin{pmatrix}\beta_{i1}\\ \beta_{i2}\\ \vdots\\ \beta_{in}\end{pmatrix}$.

$= \beta_{i1}\vec{\alpha_1} + \beta_{i2}\vec{\alpha_2} + \cdots + \beta_{in}\vec{\alpha_n}$

$\therefore A\vec{\beta_i}$ is the linear combination of all the columns of A.

$\forall i = 1, 2, \cdots k$).

$\therefore A\vec{\beta_i} \in C(A)$.

$\because C(AB)$ is spanned by all $A\vec{\beta_i}$ $(i = 1, 2, \cdots n)$.

$\therefore C(AB) \subseteq C(A)$.

2. a) $A = (\vec{\beta_1}, \vec{\beta_2}, \vec{\beta_3})$

$\because \vec{\beta_2} = 2\vec{\beta_1}$

$\therefore C(A)$ is spanned by $\vec{\beta_1}$ and $\vec{\beta_3}$.

$\therefore C(A)$ is a plane:

$\{\vec{v} \mid \vec{v} = c_1\vec{\beta_1} + c_2\vec{\beta_3}, c_1, c_2 \in R\}$.

b) $\because \begin{pmatrix}1\\1\\1\end{pmatrix} \neq c_1\begin{pmatrix}1\\0\\2\end{pmatrix} + c_2\begin{pmatrix}1\\2\\3\end{pmatrix}$.

for $\forall c_1, c_2 \in R$.

$\therefore C(A)$ is R^3.

(c). $C(A\,AB) = C(A)$.

($\because$ every column of AB is in A).

$\therefore$ It has no relationship with B,

$\because \begin{pmatrix}1\\1\\1\end{pmatrix} \neq c_1\begin{pmatrix}1\\0\\2\end{pmatrix} + c_2\begin{pmatrix}1\\2\\3\end{pmatrix}$

for $\forall c_1, c_2 \in R$.

$\therefore C(A) = R^3$.

3. $A = (\vec{\beta_1}, \vec{\beta_2} \cdots \vec{\beta_n})$

$\therefore kA = (k\vec{\beta_1}, k\vec{\beta_2} \cdots k\vec{\beta_n})$

$\therefore$ each column of kA is a linear combination of columns in A.

$\therefore C(kA) \subseteq C(A)$. ①

Based on the same reason, each column of A is also a linear combination of columns in kA.

$(\vec{\beta_i} = \frac{1}{k} \cdot (k\vec{\beta_i}))$.

$\therefore C(A) \subseteq C(kA)$ ②

$\because$ ① and ②

$\therefore C(A) = C(kA)$.

4. $\because C(AB) \subseteq C(A)$ $\quad A_{m\times n}$

for any $B_{n\times k}$.

Set $B = A^{-1}$ when A is invertible

$\therefore C(AA^{-1}) \subseteq C(A)$

$C(I) \subseteq C(A)$.

$\because C(I) = R^n$,

$\therefore R^n \subseteq C(A)$. ①

And $\because$ every column of A is from R^n,

$\therefore C(A) \subseteq R^n$. ②

$\therefore$ ① and ② $C(A) = R^n$.

5. $\because$ all columns of A come from R^m.

$\therefore C(A) \subseteq R^m$.

$\because A$ just has n columns and R^n needs at least n columns to be spanned.

$\therefore C(A) \subseteq R^n$

$\therefore C(A)$ is the subspace of R^k $(k = \min(m, n))$.