

20221115 Assignment – Column Space and Nullspace

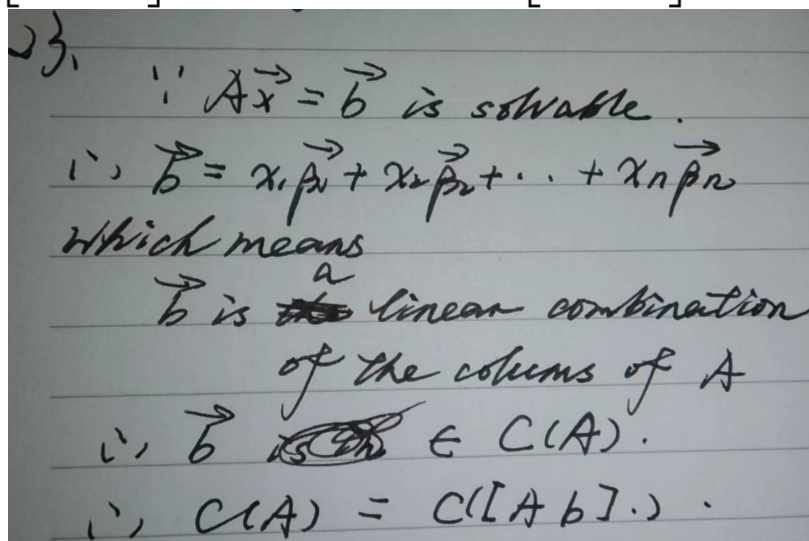
Section 3.1

19 The column space of A is the x -axis = all vectors $(x, 0, 0)$: a *line*. The column space of B is the xy plane = all vectors $(x, y, 0)$. The column space of C is the line of vectors $(x, 2x, 0)$.

20 (a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + b_3$ in equation 3: Solution only if $b_3 = -b_1$.

23 The extra column \mathbf{b} enlarges the column space unless \mathbf{b} is *already in* the column space.

$$[A \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \text{(larger column space)} \\ \text{(no solution to } Ax = \mathbf{b}) \end{array} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{l} \text{(\mathbf{b} is in column space)} \\ \text{(} Ax = \mathbf{b} \text{ has a solution)} \end{array}$$



24 The column space of AB is *contained in* (possibly equal to) the column space of A . The example $B =$ zero matrix and $A \neq 0$ is a case when $AB =$ zero matrix has a smaller column space (it is just the zero space \mathbf{Z}) than A .

26 The column space of any invertible 5 by 5 matrix is \mathbf{R}^5 . The equation $A\mathbf{x} = \mathbf{b}$ is always solvable (by $\mathbf{x} = A^{-1}\mathbf{b}$) so every \mathbf{b} is in the column space of that invertible matrix.

27 (a) *False*: Vectors that are *not* in a column space don't form a subspace.

(b) *True*: Only the zero matrix has $C(A) = \{\mathbf{0}\}$. (c) *True*: $C(A) = C(2A)$.

(d) *False*: $C(A - I) \neq C(A)$ when $A = I$ or $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (or other examples).

28 $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ do not have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $C(A)$. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ has $C(A) = \text{line in } \mathbf{R}^3$.

31 If $S = C(A)$ and $T = C(B)$ then $S + T$ is the column space of $M = [A \ B]$.

32 The columns of AB are combinations of the columns of A . So all columns of $[A \ AB]$ are already in $C(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

For square matrices, the column space is \mathbf{R}^n exactly when A is *invertible*.

Section 3.2

5 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

11 The nullspace contains only $\mathbf{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is \mathbf{R}^5 , because we can solve $A\mathbf{x} = \mathbf{b}$ and every \mathbf{b} is in the column space.