20221115 Assignment – Column Space and Nullspace

Section3.1

19 The column space of A is the x-axis $=$ all vectors $(x, 0, 0)$: a *line*. The column space of B is the xy plane $=$ all vectors $(x, y, 0)$. The column space of C is the line of vectors $(x, 2x, 0)$.

20 (a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + b_3$ in equation 3: Solution only if $b_3 = -b_1$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.

$$[A \;\; \boldsymbol{b}] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix} \begin{matrix} \text{(larger column space)} \\ \text{(no solution to } A\boldsymbol{x} = \boldsymbol{b}\text{)} \end{matrix} \quad \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix} \begin{matrix} (\boldsymbol{b} \text{ is in column space}) \\ (A\boldsymbol{x} = \boldsymbol{b} \text{ has a solution}) \end{matrix}$$

23.

∵ $A\vec{x} = \vec{b}$ is solvable.

∴ $\vec{b} = x_1\vec{p_1} + x_2\vec{p_2} + \cdots + x_n\vec{p_n}$

Which means

$\vec{b}$ is ~~the~~ a linear combination of the colums of A

∴ $\vec{b} \in C(A)$.

∴ $C(A) = C([A\ b])$.

24 The column space of AB is *contained in* (possibly equal to) the column space of A. The example $B =$ zero matrix and $A \neq 0$ is a case when $AB =$ zero matrix has a smaller column space (it is just the zero space $\mathbf{Z}$) than A.

26 The column space of any invertible 5 by 5 matrix is $\mathbf{R}^5$. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is always solvable (by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$) so every $\boldsymbol{b}$ is in the column space of that invertible matrix.

27 (a) *False*: Vectors that are *not* in a column space don't form a subspace.
(b) *True*: Only the zero matrix has $\boldsymbol{C}(A) = \{\mathbf{0}\}$. (c) *True*: $\boldsymbol{C}(A) = \boldsymbol{C}(2A)$.
(d) *False*: $\boldsymbol{C}(A - I) \neq \boldsymbol{C}(A)$ when $A = I$ or $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (or other examples).

28 $A = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ do not have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\boldsymbol{C}(A)$. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ has $\boldsymbol{C}(A) =$ line in $\mathbf{R}^3$.

31 If $\boldsymbol{S} = \boldsymbol{C}(A)$ and $\boldsymbol{T} = \boldsymbol{C}(B)$ then $\boldsymbol{S} + \boldsymbol{T}$ is the column space of $M = [\,A \quad B\,]$.

32 The columns of AB are combinations of the columns of A. So all columns of $[\,A \quad AB\,]$ are already in $\boldsymbol{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ exactly when A is *invertible*.

Section3.2

5 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

11 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.