Quiz 5-6

1. (1) $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$.

(2) $n=2$. $\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}^2$

$= \begin{pmatrix}\cos^2\phi-\sin^2\phi & -2\sin\phi\cos\phi\\ 2\sin\phi\cos\phi & \cos^2\phi-\sin^2\phi\end{pmatrix}$

$= \begin{pmatrix}\cos(2\phi) & -\sin(2\phi)\\ \sin(2\phi) & \cos(2\phi)\end{pmatrix}$.

$n=3$

$\begin{pmatrix}\cos(2\phi) & -\sin(2\phi)\\ \sin(2\phi) & \cos(2\phi)\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$

$= \begin{pmatrix}\cos(3\phi) & -\sin 3\phi\\ \sin 3\phi & \cos(3\phi)\end{pmatrix}$.

$\therefore \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}^n = \begin{pmatrix}\cos(n\phi) & -\sin(n\phi)\\ \sin(n\phi) & \cos(n\phi)\end{pmatrix}$.

(3) $A = \begin{pmatrix}1 & 2 & 1\\ 0 & -1 & 2\\ 3 & 3 & 3\end{pmatrix}$

$A^T = \begin{pmatrix}1 & 0 & 3\\ 2 & -1 & 3\\ 1 & 2 & 3\end{pmatrix}$.

$\left(\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\ 0 & -1 & 2 & 0 & 1 & 0\\ 3 & 3 & 3 & 0 & 0 & 1\end{array}\right)$

$\to \left(\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\ 0 & -1 & 2 & 0 & 1 & 0\\ 0 & -3 & 0 & -3 & 0 & 1\end{array}\right)$.

$\left(\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\ 0 & -1 & 2 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & -\frac{1}{3}\end{array}\right)$.

$\to \left(\begin{array}{ccc|ccc}1 & 0 & 1 & -1 & 0 & \frac{2}{3}\\ 0 & 0 & 2 & 1 & 1 & -\frac{1}{3}\\ 0 & 1 & 0 & 1 & 0 & -\frac{1}{3}\end{array}\right)$.

$\to \left(\begin{array}{ccc|ccc}1 & 0 & 1 & -1 & 0 & \frac{2}{3}\\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{6}\\ 0 & 1 & 0 & 1 & 0 & -\frac{1}{3}\end{array}\right)$.

$\to \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & \frac{5}{6}\\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{6}\\ 0 & 1 & 0 & 1 & 0 & -\frac{1}{3}\end{array}\right)$.

$\to \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & \frac{5}{6}\\ 0 & 1 & 0 & 1 & 0 & -\frac{1}{3}\\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{6}\end{array}\right)$.

$\therefore A^{-1} = \begin{pmatrix}-\frac{3}{2} & -\frac{1}{2} & \frac{5}{6}\\ 1 & 0 & -\frac{1}{3}\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{6}\end{pmatrix}$.

2. (1) $3! = 6$.

$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$,

$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$, $\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$, $\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$

(2) P is invertible

$\therefore P^{-1}P^5 = P^{-1}P$

$P^4 = I$.

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∴ P can be $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

3. (1) $X^T = ((-A)A^T)^T$
$= (A^T)^T(-A)^T$
$= A(-A^T)$
$= -AA^T = X$.

∴ X is symmetric

(2) $XY = -AA^TA^TA$.

∴ $(XY)^T = Y^TX^T$
$= (A^TA)^T(-AA^T)^T$
$= A^TA(-AA^T)$
$= -A^TAAA^T$

$(XY)^T \neq XY$.

∴ not symmetric.

4. (1) $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

∴ $P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$P^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$.

(2) It's obvious that

$P\vec{v} = \vec{v}$ when $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(3) ∵ $P^3 = I$

∴ If $\vec{v}$ is ~~rotaced~~ for 3 times, it goes back to the original status.

∴ the angle of rotation is $120°$.

(4) $\cos\theta = \dfrac{\vec{v}\cdot(P\vec{v})}{\|\vec{v}\|\cdot\|P\vec{v}\|}$.

$= \dfrac{16+15-5\sqrt{14}-3\sqrt{14}}{16+9+25+14}$

$= \dfrac{31-8\sqrt{14}}{64}$

∴ $\theta = \arccos\dfrac{31-8\sqrt{14}}{64}$

5. (1) $\vec{x}^T\vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

$\vec{y}^T\vec{x} =$

$\vec{x}\vec{y}^T = \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ \vdots & x_2y_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ x_ny_1 & \cdots & \cdots & x_ny_n \end{pmatrix}$.

∴ $Tr(\vec{x}\vec{y}^T) = \sum_{i=1}^{n} x_iy_i = Tr(\vec{y}^T\vec{x})$.

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(2) Set $A_{m\times n}$, $B_{n\times m}$.

$\because AB = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}$

$= \begin{pmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \cdots & \sum_{i=1}^{n} a_{1i}b_{im} \\ \vdots & & \vdots \\ \sum_{i=1}^{n} a_{mi}b_{i1} & \cdots & \sum_{i=1}^{n} a_{mi}b_{im} \end{pmatrix}$

$\therefore Tr(AB) = \sum_{i=1}^{n} a_{1i}b_{i1} + \cdots + \sum_{i=1}^{n} a_{mi}b_{im}$

$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji}b_{ij} \right)$

$= \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ji}b_{ij}$

$BA = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

$= \begin{pmatrix} \sum_{i=1}^{m} b_{1i}a_{i1} & \cdots & \sum_{i=1}^{m} b_{1i}a_{in} \\ \vdots & & \vdots \\ \sum_{i=1}^{m} b_{ni}a_{i1} & \cdots & \sum_{i=1}^{m} b_{ni}a_{in} \end{pmatrix}$.

$\therefore Tr(BA) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} b_{ji}a_{ij} \right)$

$= \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji}a_{ij}$

$\therefore Tr(AB) = Tr(BA)$

(3) $A = 2$. $B = (3\ 3)$

$C = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\because S = -CA^{-1}B + D$

$= -\begin{pmatrix} 4 \\ 4 \end{pmatrix} (2^{-1}) (3\ 3) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -6 & -6 \\ -6 & -6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -6 \\ -6 & -5 \end{pmatrix}$.

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1. $\left(\begin{array}{cc|cc} A & 0 & I & 0 \\ C & B & 0 & I \end{array} \right)$.

$= \left(\begin{array}{cc|cc} A & 0 & I & 0 \\ 0 & B & -CA^{-1} & I \end{array} \right)$.

$= \left(\begin{array}{cc|cc} A & 0 & I & 0 \\ 0 & I & -B^{-1}CA^{-1} & B^{-1} \end{array} \right)$.

$= \left(\begin{array}{cc|cc} I & 0 & A^{-1} & 0 \\ 0 & I & -B^{-1}CA^{-1} & B^{-1} \end{array} \right)$.

$\therefore$ It doesn't matter

2. (1) $S = -CA^{-1}B + D$.

(2) $\left(\begin{array}{cc|cc} A & B & I & 0 \\ 0 & S & 0 & I \end{array} \right)$.

$\left(\begin{array}{cc|cc} A & 0 & I & -A^{-1}B \\ 0 & S & 0 & I \end{array} \right)$.

$\left(\begin{array}{cc|cc} I & 0 & A^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ 0 & I & 0 & (D-CA^{-1}B)^{-1} \end{array} \right)$

3.

4.

3.(2) $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$.

$A_1 = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$. $A_2 = 0$

$A_3 = 0$. $A_4 = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$.

$\therefore A = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix}$.

$\therefore A^{-1} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & A_4^{-1} \end{pmatrix}$.

A_4^{-1}: $\left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$.

$\rightarrow \left(\begin{array}{ccc|ccc} 0 & 4 & 3 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right)$.

$\rightarrow \left(\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & -6 & -4 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right)$.

$\rightarrow \left(\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & -6 & -4 \\ 1 & 0 & 0 & 1 & -4 & -3 \\ 0 & 1 & 0 & 1 & -5 & -3 \end{array}\right)$.

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & -3 \\ 0 & 1 & 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & -1 & 6 & 4 \end{array}\right)$.

A_1^{-1}:

$\because \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$.

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right)$.

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -3 & 3 & -2 & 1 \end{array}\right)$.

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right)$.

$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right)$.

$\therefore A_1^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ -1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$.

$\therefore A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -1 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & -3 \\ 0 & 0 & 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & -1 & 6 & 4 \end{pmatrix}$